Note: These lecture notes are accompanied with video animations in a separate pdf file.

Flows on the circle

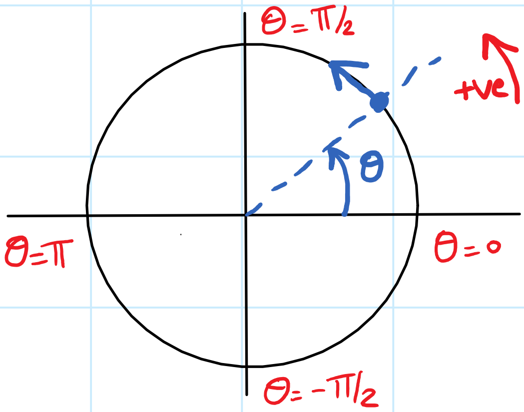
So far $\longrightarrow$ $\dot{x} = f(x)$ $x \in \mathbb{R}$

No oscillations!

However...

Consider : $\dot{\theta} = f(\theta)$ θ evolves on a circle $\theta \in]-\pi, \pi]$

vector field on the circle



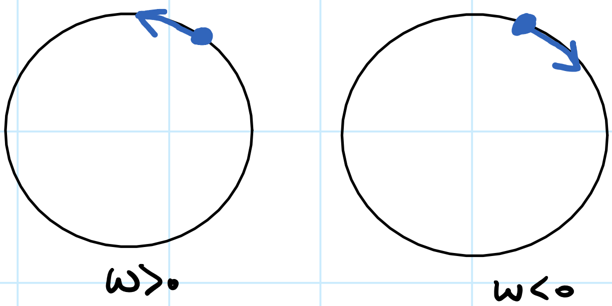
These models are simplest way to model oscillations (periodic orbits)

Simplest Example : Uniform Oscillator

$$\begin{cases} \dot{\theta} = \omega \\ \theta(0) = \theta_0 \end{cases}$$

(ω : constant)

$$\theta(t) = \omega t + \theta_0$$



Assume $\omega > 0$ period T : $\theta(t+T) = \theta(t) + 2\pi$

$$\omega(\cancel{t}+T) + \cancel{\theta_0} = \cancel{\omega t} + \cancel{\theta_0} + 2\pi$$

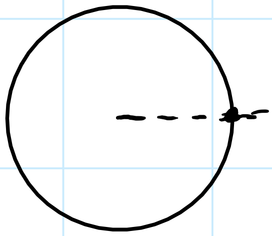
$$\Rightarrow T = \frac{2\pi}{\omega}$$

Figure 1 shows an animation for both $\omega > 0$ and $\omega < 0$

Weird Example: $S' \begin{cases} \dot{\theta} = f(\theta) \\ \theta(0) = 0 \end{cases}$, $f(\theta) = \theta$

Question: can system S' be regarded as vector field on a circle?

Answer: No ⚠



$\theta = 0 \longrightarrow f(0) = 0$

$\theta = 2\pi \longrightarrow f(2\pi) = 2\pi$ $\neq$

multiple values for $f(\theta)$ on the same location on the circle!

Fact: vector fields on circles have to satisfy

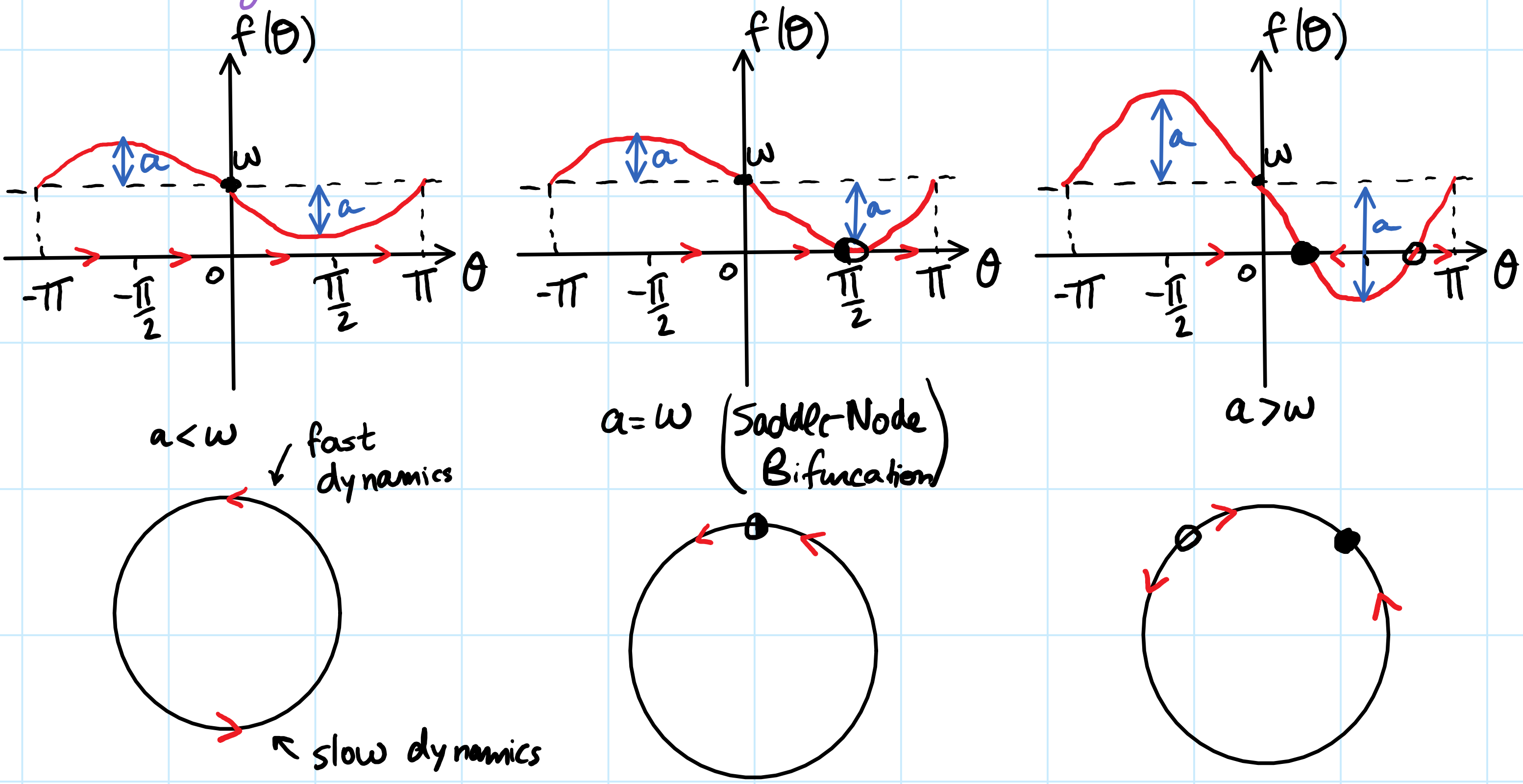
$$f(\theta + 2\pi) = f(\theta) \quad \text{"}2\pi\text{-peridic functions"}$$

More interesting Example: (Non-uniform Oscillator)

$$\begin{cases} \dot{\theta} = \omega - a\sin\theta =: f(\theta) + \text{constant} \\ \theta(0) = \theta_0 \end{cases}$$

aside: Observe that $f(\theta + 2\pi) = f(\theta)$ ✓

Figure 2 shows an animation of $f(\theta)$ as a is varied.



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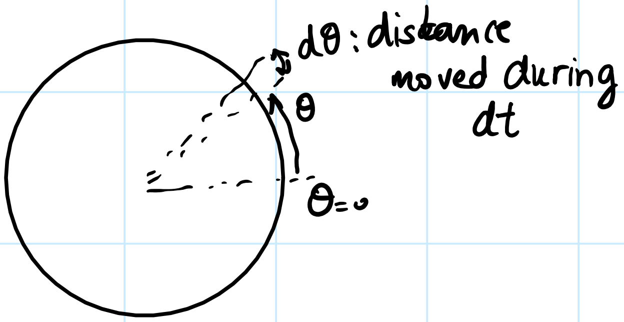
Figures 3, 4 and 5 show animations for the cases: $a<\omega$, $a=\omega$, and $a>\omega$, respectively.

Period T:

$$\dot{\theta} = \omega - a\sin\theta$$

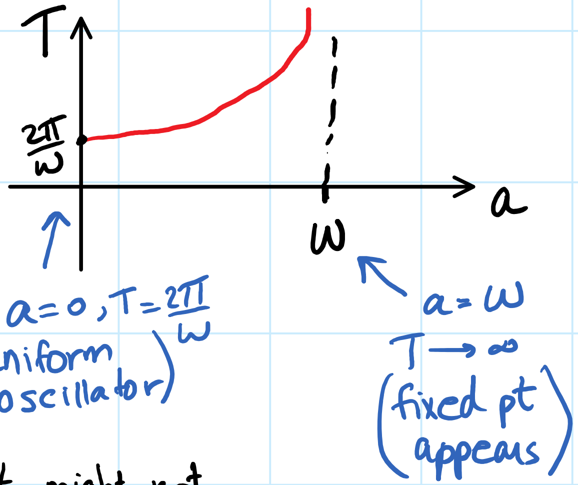


$$dt = \frac{\text{distance}}{\text{velocity}} = \frac{d\theta}{\dot{\theta}}$$

$$T = \int dt = \int_{\theta=0}^{\theta=2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$

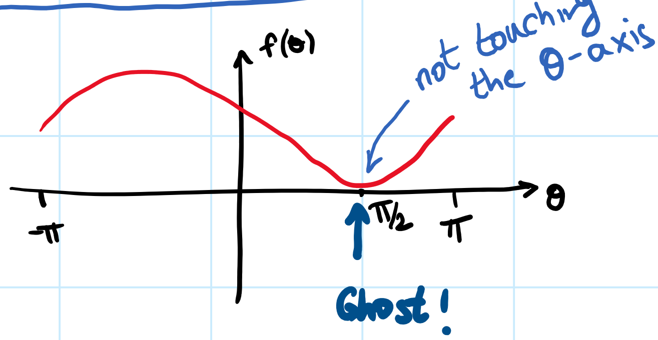
$$\Rightarrow T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

trick: $u = \tan\left(\frac{\theta}{2}\right)$ (change of variable)

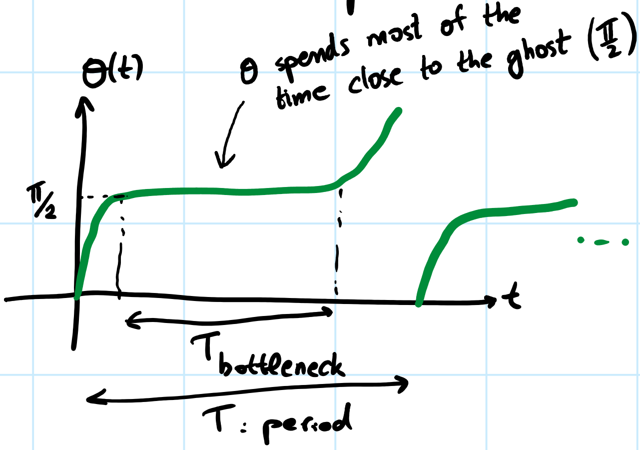


For more complicated dynamical systems, it might not be possible to find an exact formula for the period T.

Ghosts & Bottlenecks:



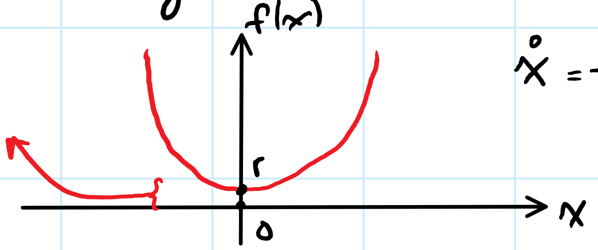
"invisible fixed pt"
fixed pt is not there but you can feel it!



Argue that $T \approx T_{bottleneck}$

Figure 6 shows an animation that illustrates how a ghost affects the dynamics.

General case: Dynamics with ghosts can be approximated by normal forms of Saddle-node bifurcation



$\dot{x} = f(x)$; $f(x) = r + x^2$

$T_{bottleneck} \approx$ time for x to go from $-\infty \longrightarrow +\infty$

$$= \int dt = \int_{-\infty}^{+\infty} \frac{dx}{\dot{x}} = \int_{-\infty}^{+\infty} \frac{dx}{r+x^2} = \frac{\pi}{\sqrt{r}}$$

$$\boxed{\text{for } \dot{x} = r + x^2 \qquad T_{bottleneck} = \frac{\pi}{\sqrt{r}} \qquad (r \text{ is small})}$$

For our example: $\dot{\theta} = \omega - a\sin\theta \qquad (a \approx \omega)$

Goal: write it in a form that looks like "$\dot{x} = r + x^2$"

$\rightarrow$ Have to describe the dynamics close to the ghost (i.e. $\pi/2$)

Introduce $\phi := \theta - \pi/2$

$$\frac{d\phi}{dt} = \frac{d\theta}{dt} = \omega - a\sin\theta = \omega - a\sin(\phi + \pi/2) = \omega - a\cos\phi$$

$$= \omega - a\left(1 - \frac{\phi^2}{2} + O(\phi^3)\right) \approx (\omega - a) + \frac{a}{2}\phi^2$$

$\uparrow$ Taylor Expansion ($\phi \approx 0$)

$$\therefore \frac{d\phi}{dt} = (\omega - a) + \frac{a}{2}\phi^2 \xrightarrow[\substack{x := \sqrt{\frac{a}{2}}\,\phi \\ \tau := \sqrt{\frac{a}{2}}\,t}]{r := \omega - a} \frac{dx}{d\tau} = r + x^2$$

Normal Form of Saddle-Node Bifurcation

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$$\Rightarrow T_{\text{bottleneck}} \text{ (in the units of } \tau) = \frac{\pi}{\sqrt{r}}$$

$$\Rightarrow T_{\text{bottleneck}} \text{ (in the units of } t) = \sqrt{\frac{2}{a}} \cdot T_{\text{bottleneck}} \text{ (in the units of } \tau)$$

$$= \sqrt{\frac{2}{a}} \cdot \frac{\pi}{\sqrt{r}} = \sqrt{\frac{2}{a}} \cdot \frac{\pi}{\sqrt{\omega - a}}$$

$$\therefore \boxed{T_{\text{bottleneck}} = \sqrt{\frac{2}{a}} \frac{\pi}{\sqrt{\omega - a}} \quad (\omega \approx a)}$$

$\Rightarrow$ Can argue that the Period $\approx T_{\text{bottleneck}}$

Note: This approximation is valid only when $\omega \approx a$ ⚠

Final Example: Fireflies

youTube video:

[Watch: Fireflies Glowing in Sync to Attract Mates | National Geographic]

Model:

θ_1 : phase of the flash light

θ_2 : phase of the firefly.

$$\begin{cases} \dot{\theta}_1 = \Omega & (\Omega : \text{constant}) \\ \dot{\theta}_2 = \omega + A\sin(\theta_1 - \theta_2) \end{cases}$$

Note: • when $\theta_1 > \theta_2$ (flashlight is ahead of the firefly)

$\Rightarrow \sin(\theta_1 - \theta_2) > 0$

$$\Rightarrow \dot{\theta}_2 = \omega + \underbrace{A\sin(\theta_1 - \theta_2)}_{>0}$$

(further the push to θ_2 to speed it up so it follows the flashlight)

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- when $\theta_1 < \theta_2$ (flashlight is behind the firefly)

$$\Rightarrow \sin(\theta_1 - \theta_2) < 0$$

$$\Rightarrow \dot{\theta}_2 = \omega + \underbrace{A\sin(\theta_1-\theta_2)}_{\hat{0}}$$

(-ve push to θ_2 to slow it down so it follows the flashlight)

Figures 7, 8, and 9 show simulations that describe the evolution of $\theta_1(t)$ and $\theta_2(t)$ for three cases

Figure 7 $\longrightarrow$ $\Omega = \frac{\pi}{2}$; $\omega = \frac{\pi}{2}$; $A = \frac{\pi}{7}$

Figure 8 $\longrightarrow$ $\Omega = \frac{2\pi}{3}$; $\omega = \frac{\pi}{2}$; $A = \frac{\pi}{5}$

Figure 9 $\longrightarrow$ $\Omega = \frac{2\pi}{3}$; $\omega = \frac{\pi}{2}$; $A = \frac{\pi}{7}$

Analysis: $\phi := \theta_1 - \theta_2$ (phase difference)

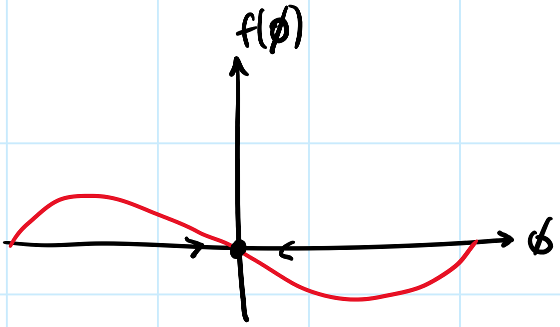
dynamics of ϕ : $\frac{d\phi}{dt} = \frac{d\theta_1}{dt} - \frac{d\theta_2}{dt} = \Omega - (\omega + A\sin(\underbrace{\theta_1-\theta_2}_{\phi}))$

$$\boxed{\frac{d\phi}{dt} = (\Omega - \omega) - A\sin\phi}$$

$$\Big\downarrow \; \tau := At, \quad \mu := \frac{\Omega-\omega}{A}$$

$$\boxed{\frac{d\phi}{d\tau} = \mu - \sin\phi =: f(\phi)}$$

$$\frac{d\phi}{d\tau} = \mu - \sin\phi =: f(\phi)$$

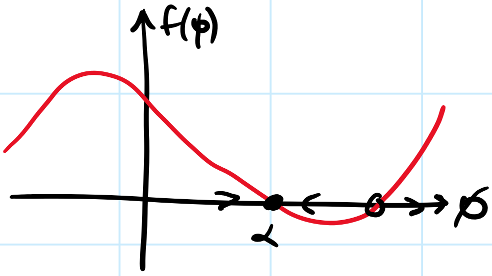


(See Figure 7)

$\mu = 0$

$\mu := \frac{\Omega - W}{A} \Rightarrow \Omega = W$

"synchrony"
stable fixed pt is $\phi = 0$
phase difference

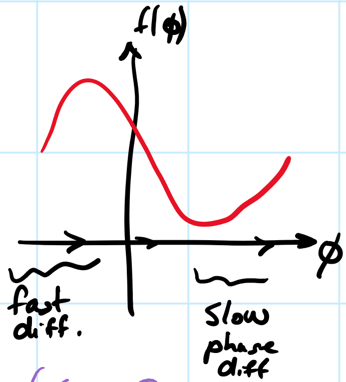


(See Figure 8)

$0 < \mu < 1$

fixed pt $\phi = \alpha$

phase difference is constant $= \alpha$
(no synchrony though)
"phase locked"



(See Figure 9)

$\mu > 1$

no fixed pts
"phase drift"

The range of entrainment is the range of driving angular frequency (i.e. frequency of flash light) for which the firefly can either synchronize or phase lock.

This is possible for $-1 < \mu < 1$

$$\Rightarrow -1 < \frac{\Omega - W}{A} < 1$$

$$\Rightarrow \boxed{W - A < \Omega < W + A}$$